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Ageing without detailed balance in the bosonic contact and pair-contact processes: exact results

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Abstract

Ageing in systems without detailed balance is studied in the exactly solvable bosonic contact process and the critical bosonic pair-contact process. The two-time correlation function and the two-time response function are explicitly found. In the ageing regime, the dynamical scaling of these is analysed and exact results for the ageing exponents and the scaling functions are derived. For the critical bosonic pair-contact process, the autocorrelation and autoresponse exponents agree but the ageing exponents a and b are shown to be distinct.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Systems brought rapidly out of an initial state into a region in parameter space which is characterized by several competing stationary states may undergo ageing behaviour. Dynamical scaling and universality was first noticed in the mechanical properties of several glass-forming systems rapidly quenched into their glassy phase [1] and has since been found and studied intensively in a large variety of systems relaxing towards an equilibrium state, see [2–8] for recent reviews. In what follows we shall restrict to systems without any macroscopic conservation laws and to systems without frustrations, conditions which are paradigmatically met in simple ferromagnets with dynamics which satisfy detailed balance.

Convenient tools for the study of ageing behaviour in such systems [9] are the two-time autocorrelation and autoresponse functions, which in the *ageing regime* $t, s \gg 1$ and $t - s \gg 1$ are expected to show the scaling behaviour

$$C(t, s) = \langle \phi(t)\phi(s) \rangle \sim s^{-b} f_C(t/s) \tag{1}$$

$$R(t,s) = \left.\frac{\delta\langle\phi(t)\rangle}{\delta h(s)}\right|_{h=0} \sim s^{-1-a} f_R(t/s). \tag{2}$$

Here $\phi(t)$ is the order parameter and $h(s)$ is the conjugate field, t is called the observation time and s is the waiting time. For large arguments $y \to \infty$, one generically expects

$$f_C(y) \sim y^{-\lambda_C/z}, \qquad f_R(y) \sim y^{-\lambda_R/z} \tag{3}$$

where λ_C and λ_R, respectively, are known as autocorrelation [10, 11] and autoresponse exponents [12]. While the ageing exponents a and b can be expressed in terms of the dynamical exponent z and equilibrium exponents, the exponents $\lambda_{C,R}$ are independent of these but related to the so-called initial slip exponents [13]. For critical quenches

$$a = b = \frac{2\beta}{\nu z}, \qquad \text{at} \quad T = T_c \tag{4}$$

(where β and ν are the usual equilibrium critical exponents) is a consequence of the fluctuation-dissipation theorem and of time-translation invariance in the scale-invariant equilibrium steady state.

Turning to the scaling functions, it has been suggested [14, 15] that their form could be determined from the requirement of covariance under so-called *local* scale transformations which are constructed from the requirement of including the special conformal transformations $t \mapsto (\alpha t + \beta)/(\gamma t + \delta)$ with $\alpha\delta - \beta\gamma = 1$ in time. For the response function this leads to [14–16]

$$R(t,s) = s^{-1-a} f_R(t/s), \qquad f_R(y) = f_0 y^{1+a'-\lambda_R/z}(y-1)^{-1-a'} \tag{5}$$

where a' is a new independent exponent and f_0 is a normalization constant. This form (or equivalently an integrated response) describes very well all available numerical data for ferromagnetic systems quenched to $T < T_c$ [14, 15, 17] and was shown to be exact in several exactly solvable models, see [5] and references therein. In these systems, $z = 2$ [2]. On the other hand, for quenches to $T = T_c$, one has in general $z \neq 2$, but the agreement of (5) with numerical data is still almost perfect [14, 18, 19]. However, a second-order ε-expansion calculation in the O(n)-symmetric ϕ^4-field theory produced a small but systematic correction with respect to (5) [8], and there is support of this finding from a numerical study of $R(t,s)$ in momentum space [20]. While in most systems studied so far, $a = a'$ holds true; several models with $a \neq a'$ are also known [16].

The approach of local scale invariance uses the dynamical symmetry of deterministic equations, whereas the influence of noise is essential in the understanding of non-equilibrium dynamics. However, considering a description of the ageing system in terms of a Langevin equation, it can be shown that *if* the deterministic (i.e. noiseless) part satisfies local scale invariance, then (i) the response function $R(t,s)$ is noise independent and (ii) the autocorrelator $C(t,s)$ can be reduced to noiseless three- and four-point response functions [21]. A further extension of local scale transformation to include the diffusion constant as a new dynamical variable then permits the determination of $C(t,s)$ and the result was found to be in good agreement with simulational data in the 2D Ising model quenched to $T < T_c$ [22].

An important ingredient in the ageing studies discussed so far is the assumption of detailed balance for the dynamics. This begs the question what might happen if that condition is relaxed. Indeed, numerical studies of the contact process, the simplest system of that kind, gave the following results [23, 24].

(i) Dynamical scaling and ageing only occur *at* the critical point. This is expected since both inside the active and the inactive phases there merely is a single stable stationary state.

(ii) At criticality, the scaling forms equations (1) and (2) hold true for the (connected) autocorrelator and the response function, but with the scaling relation

$$a + 1 = b = \frac{2\beta}{\nu_\perp z}, \tag{6}$$

in contrast to equation (4), with β and $\nu_\perp$ being now standard steady-state exponents.

In order to get a better understanding of these results, it would be helpful to study the ageing behaviour in exactly solvable but non-trivial models without detailed balance. Remarkably, it has been realized by Houchmandzadeh [25] and by Paessens and Schütz [26] that the bosonic versions of the contact process and of the critical pair-contact process (where an arbitrary number of particles are allowed on each lattice site) are exactly solvable, at least to the extent that the dynamical scaling behaviour of equal-time correlators can be analysed exactly [25–27]. Here we extend their work by means of an exact calculation of the two-time correlation and response functions for the bosonic contact process and the critical bosonic pair-contact process. In section 2, we define the models and write down the closed systems of equations of motion for the correlation and response functions. We also recall the existing results on the single-time correlators [25, 26]. In section 3, we discuss the bosonic contact process and in section 4, we describe our results for the critical pair-contact process. In section 5 we give the results for the two-time response functions. As we shall see, the critical bosonic pair-contact process provides a further example of a model where a and b are different. In section 6 we conclude. A detailed discussion of local scale invariance in these models will be presented in a following paper.

2. The models

Consider the following stochastic process: on an infinite d-dimensional hypercubic lattice, particles move diffusively with rate D in each spatial direction. Each site may contain an arbitrary non-negative number of particles. Furthermore, on any given site the following reactions for the particles A are allowed

$$\begin{aligned} mA &\longrightarrow (m+k)A; &\qquad \text{with rate } \mu \\ pA &\longrightarrow (p-\ell)A; &\qquad \text{with rate } \lambda. \end{aligned} \tag{7}$$

It is to be understood that on a given site, out of any set of m particles k additional particles are created with rate μ and ℓ particles are destroyed out of any set of $p \geqslant \ell$ particles with rate λ. Diffusion applies on single particles. We shall be concerned with two special cases:

(i) the *bosonic contact process*, where $p = m = 1$, hence $\ell = 1$. The value of k is unimportant and will be fixed to $k = 1$ as well.
(ii) the *bosonic pair-contact process*, where $p = m = 2$.

While the bosonic contact process arose from a study on the origin of clustering in biology [25], the bosonic pair-contact process as defined here [26] is an offshoot of a continuing debate about the critical behaviour of the diffusive pair-contact process (PCPD), see [28] for a recent review. Initially, this model was introduced [29] in an attempt to understand the meaning of 'imaginary' versus 'real' noise but the associated field theory turned out to be unrenormalizable [29, 30]. A lattice version (with the 'fermionic' constraint of not more than one particle per site) of the model contains the reactions $2A \to \emptyset$ and $2A \to 3A$ together with single-particle diffusion $A\emptyset \leftrightarrow \emptyset A$ and was first studied numerically in [31]. An intense debate on the universality class of this model followed, see [28], and several mutually exclusive conclusions on the critical behaviour continue to be drawn, see [30, 32–36] for recent work. The bosonic pair-contact process has a dynamic exponent $z = 2$ [26] and is hence distinct from the PCPD where $z < 2$. Its study will not so much shed light on any open question concerning the PCPD, but it should rather be viewed as a non-trivial example of an exactly solvable non-equilibrium many-body system to be studied in its own right.

The master equation is written in a quantum Hamiltonian formulation as $\partial_t |P(t)\rangle = -H|P(t)\rangle$ [37, 38] where $|P(t)\rangle$ is the time-dependent state vector and the Hamiltonian H can be expressed in terms of annihilation and creation operators $a(\boldsymbol{x})$ and $a^\dagger(\boldsymbol{x})$. We also define the particle number operator as $n(\boldsymbol{x}) = a^\dagger(\boldsymbol{x})a(\boldsymbol{x})$. Then the Hamiltonian of model (7) reads [26]

$$\begin{aligned} H = -D\sum_{r=1}^{d}\sum_{\boldsymbol{x}}[a(\boldsymbol{x})a^\dagger(\boldsymbol{x}+\boldsymbol{e}_r) + a^\dagger(\boldsymbol{x})a(\boldsymbol{x}+\boldsymbol{e}_r) - 2n(\boldsymbol{x})] \\ -\lambda\sum_{\boldsymbol{x}}\left[(a^\dagger(\boldsymbol{x}))^{p-\ell}(a(\boldsymbol{x}))^p - \prod_{i=1}^{p}(n(\boldsymbol{x}) - i + 1)\right] \\ -\mu\sum_{\boldsymbol{x}}\left[(a^\dagger(\boldsymbol{x}))^{m+k}(a(\boldsymbol{x}))^m - \prod_{i=1}^{m}(n(\boldsymbol{x}) - i + 1)\right] - \sum_{\boldsymbol{x}} h(\boldsymbol{x}, t)a^\dagger(\boldsymbol{x}) \end{aligned} \qquad (8)$$

where $\boldsymbol{e}_r$ is the rth unit vector. For later use in the calculation of response functions, we have also added an external field which describes the spontaneous creation of a single particle $\emptyset \to A$ with a site-dependent rate $h = h(\boldsymbol{x}, t)$ on the site $\boldsymbol{x}$.

Single-time observables $g(\boldsymbol{x}, t)$ can be obtained from the time-independent quantities $g(\boldsymbol{x})$ by switching to the Heisenberg picture. They satisfy the usual Heisenberg equation of motion, from which the differential equations for the desired quantities can be obtained. The spacetime-dependent particle density $\rho(\boldsymbol{x}, t) := \langle a^\dagger(\boldsymbol{x}, t)a(\boldsymbol{x}, t)\rangle = \langle a(\boldsymbol{x}, t)\rangle$ satisfies

$$\frac{\partial}{\partial t}\langle a(\boldsymbol{x}, t)\rangle = D\Delta_{\boldsymbol{x}}\langle a(\boldsymbol{x}, t)\rangle - \lambda\ell\langle a(\boldsymbol{x}, t)^p\rangle + \mu k\langle a(\boldsymbol{x}, t)^m\rangle + h(\boldsymbol{x}, t) \qquad (9)$$

where we have used the shorthand

$$\Delta_{\boldsymbol{x}} f(\boldsymbol{x}) := \sum_{r=1}^{d}(f(\boldsymbol{x} - \boldsymbol{e}_r, t) + f(\boldsymbol{x} + \boldsymbol{e}_r, t) - 2f(\boldsymbol{x}, t)) \qquad (10)$$

and similar equations hold for the equal-time two-point correlation functions, see [26]. It turns out that for the bosonic contact process $p = m = 1$, these equations close for arbitrary values of the rates. On the other hand, for the bosonic pair-contact process where $p = m = 2$, a closed system of equations is only found along the critical line given by [26]

$$\ell\lambda = \mu k. \qquad (11)$$

This line separates an active phase with a formally infinite particle density in the steady state from an absorbing phase where the steady state particle density vanishes, see figure 1 for the schematic phase diagrams. In what follows, the essential control parameter is

$$\alpha := \mu k(k + \ell)/(2D). \qquad (12)$$

The physical nature of this transition becomes apparent when equal-time correlations are studied [25, 26] and can be formulated in terms of a *clustering transition*. By clustering we mean that particles accumulate on very few lattice sites while the other ones remain empty. Now, for the bosonic contact process, the behaviour along the critical line is independent of α. If $d \leqslant 2$, there is always clustering, while there is no clustering for $d > 2$. On the other

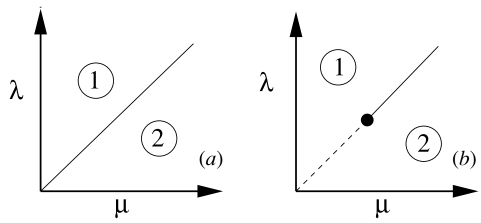

Figure 1. Schematic phase diagrams for $D \neq 0$ of (a) the bosonic contact process and the bosonic pair-contact process in $d \leqslant 2$ dimensions and (b) the bosonic pair-contact process in $d > 2$ dimensions. The absorbing region 1, where $\lim_{t\to\infty} \rho(\boldsymbol{x}, t) = 0$, is separated by the critical line equation (11) from the active region 2, where $\rho(\boldsymbol{x}, t) \to \infty$ as $t \to \infty$. On the critical line $\rho(t) := \int \mathrm{d}\boldsymbol{x}\, \rho(\boldsymbol{x}, t)$ remains constant. By varying α one moves along the critical line. Along the critical line, one may have clustering (full lines in (a) and (b)), but in the bosonic pair-contact process with $d > 2$ the steady state may also be homogeneous (broken line in (b)). These two regimes are separated by a multicritical point.

hand, in the bosonic pair-contact process, there is on the critical line a multicritical point at $\alpha = \alpha_C$, with

$$\alpha_C = \alpha_C(d) = \frac{1}{2A_1}, \qquad A_1 := \int_0^\infty \mathrm{d}u (\mathrm{e}^{-4u} I_0(4u))^d \tag{13}$$

and where $I_0(u)$ is a modified Bessel function [39], such that clustering occurs for $\alpha > \alpha_C$ only and with a more or less homogeneous state for $\alpha \leqslant \alpha_C$. Specific values are $\alpha_C(3) \approx 3.96$ and $\alpha_C(4) \approx 6.45$ and $\lim_{d\to 2} \alpha_C(d) = 0$. We are interested in studying the impact of this clustering transition on the two-time correlations and linear responses.

In order to obtain the equations of motion of the two-time correlator, the time ordering of the operators $a(\boldsymbol{x}, t)$ must be taken into account. From the Hamiltonian equation (8) without an external field h, we get the following equations of motion for the two-time correlator, after rescaling the times $t \mapsto t/(2D)$, $s \mapsto s/(2D)$, and for $t > s$ [40] (for a detailed computation, see [41]) ,

$$\frac{\partial}{\partial t}\langle a(\boldsymbol{x}, t)a(\boldsymbol{y}, s)\rangle = \frac{1}{2}\Delta_{\boldsymbol{x}}\langle a(\boldsymbol{x}, t)a(\boldsymbol{y}, s)\rangle - \frac{\lambda\ell}{2D}\langle a(\boldsymbol{x}, t)^p a(\boldsymbol{y}, s)\rangle + \frac{\mu k}{2D}\langle a(\boldsymbol{x}, t)^m a(\boldsymbol{y}, s)\rangle \tag{14}$$

which we are going to study in the next sections.

3. The bosonic contact process

For the bosonic contact process, we have $p = m = 1$, hence also $\ell = k = 1$. We first consider the critical case $\lambda\ell = \mu k$. We shall assume throughout that spatial translation invariance holds and use the notation

$$F(\boldsymbol{r}; t, s) := \langle a(\boldsymbol{x}, t)a(\boldsymbol{x} + \boldsymbol{r}, s)\rangle. \tag{15}$$

Then F satisfies a diffusion equation which is solved in a standard way by Fourier transforms. It is easy to see that the solution of the equations of motion (14) involves the single-time correlator $F(\boldsymbol{r}, t) := F(\boldsymbol{r}; t, t)$ which satisfies the equation of motion, after the usual rescaling $t \mapsto t/(2D)$ [26, equation (10)],

$$\frac{\partial}{\partial t} F(\boldsymbol{r}, t) = \Delta_{\boldsymbol{r}} F(\boldsymbol{r}, t) + \alpha\rho_0\delta_{\boldsymbol{r},\boldsymbol{0}} \tag{16}$$

and the parameter α was defined in (12). As initial conditions, we shall use throughout the Poisson distribution $F(\boldsymbol{r},0) = \rho_0^2$. Hence one arrives at the following expression of our main quantity of interest, the connected correlator[4]

$$G(\boldsymbol{r};t,s) := F(\boldsymbol{r};t,s) - \rho_0^2 = \alpha\rho_0 \int_0^s \mathrm{d}\tau\, b\left(\boldsymbol{r}, \frac{1}{2}(t+s) - \tau\right) \tag{17}$$

where ($I_r(t)$ being a modified Bessel function)

$$b(\boldsymbol{r},t) = \mathrm{e}^{-2dt} I_{r_1}(2t)\cdots I_{r_d}(2t). \tag{18}$$

We evaluate this expression in two cases.

Case 1. $\boldsymbol{r} = 0$, *t and s in the ageing regime*. In this case both s and $t-s$ are large, so that we can use the asymptotic behaviour $I_0(t) \simeq (2\pi t)^{-1/2}\,\mathrm{e}^t$ for large t [42] for the expression $b\left(\mathbf{0}, \frac{1}{2}(t+s)-\tau\right)$ under the integral in (17). We have to distinguish the cases $d>2$, $d=2$ and $d<2$. For $d>2$ we obtain

$$\begin{aligned} G(\mathbf{0},t,s) &\simeq \frac{\alpha\rho_0}{(4\pi)^{\frac{d}{2}}} \int_0^s \mathrm{d}\tau \left(\frac{1}{2}(t+s)-\tau\right)^{-\frac{d}{2}} \\ &= \frac{\alpha\rho_0}{(4\pi)^{\frac{d}{2}}\left(\frac{d}{2}-1\right)} \left(\left(\frac{t-s}{2}\right)^{-\frac{d}{2}+1} - \left(\frac{t+s}{2}\right)^{-\frac{d}{2}+1}\right). \end{aligned} \tag{19}$$

By analogy with equations (1) and (3), we expect the scaling behaviour $G(t,s) := G(\mathbf{0};t,s) = s^{-b} f_G(t/s)$. We read off the value $b = \frac{d}{2} - 1$ and the scaling function

$$f_G(y) = \frac{\alpha\rho_0}{2(2\pi)^{\frac{d}{2}}\left(\frac{d}{2}-1\right)}\left((y-1)^{-\frac{d}{2}+1} - (y+1)^{-\frac{d}{2}+1}\right). \tag{20}$$

From the expected asymptotics $f_G(y) \sim y^{-\lambda_G/z}$ for $y \gg 1$, we obtain

$$\lambda_G = d \tag{21}$$

as can be seen from the asymptotic development of (20) and where we anticipated that the dynamical exponent $z = 2$, see also [25] and below.

For $d = 2$ the integral in (19) gives a different result. We find

$$G(t,s) = f_G(t/s), \qquad f_G(y) = \frac{\alpha\rho_0}{2(2\pi)^{\frac{d}{2}}} \ln\left(\frac{y+1}{y-1}\right) \tag{22}$$

and we have the exponents $b = 0$ and $\lambda_G = 2$. The logarithmic divergence of the single-time correlator [25] reflects itself here in the logarithmic form of the scaling function.

Finally, for $d < 2$ the same computation as for $d > 2$ goes through. Now, the exponent $b = \frac{d}{2} - 1$ is *negative* which means that the two-time autocorrelator diverges, in agreement with the earlier results for the equal-time correlators in 1D [25].

Case 2: $\boldsymbol{r}$ *dependence for* $s, t-s \gg 1$. We use the asymptotic expression, valid for $u \gg 1$ and $\boldsymbol{r}^2/u$ fixed,

$$\mathrm{e}^{-dz} I_{r_1}(u)\cdots I_{r_d}(u) \simeq \frac{1}{(2\pi u)^{\frac{d}{2}}} \exp\left(-\frac{\boldsymbol{r}^2}{2u}\right) \tag{23}$$

which yields for arbitrary dimension d, when introduced into (17),

$$G(\boldsymbol{r};t,s) \simeq \frac{\alpha\rho_0}{(4\pi)^{\frac{d}{2}}}\left(\frac{\boldsymbol{r}^2}{4}\right)^{-\left(\frac{d}{2}-1\right)} \left[\Gamma\left(\frac{d}{2}-1, \frac{1}{2}\frac{\boldsymbol{r}^2}{t+s}\right) - \Gamma\left(\frac{d}{2}-1, \frac{1}{2}\frac{\boldsymbol{r}^2}{t-s}\right)\right]. \tag{24}$$

[4] In [23, 24] this same quantity was denoted by $\Gamma(t,s)$ which we avoid here in order not to create confusion with the incomplete gamma function [39].

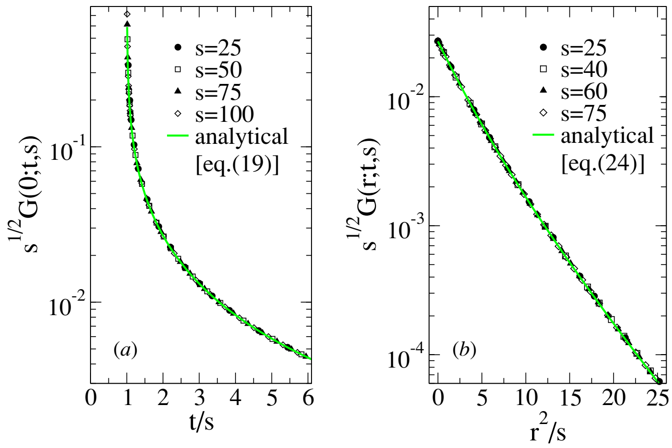

Figure 2. Scaling plots of (*a*) the autocorrelation function $G(0; t, s)$ and (*b*) the space-dependent correlation function $G(\boldsymbol{r}; t, s)$ for the critical bosonic contact process in three dimensions with $\alpha\rho_0 = 1$. In (*b*), the value of $y = t/s = 2$ was used.

The incomplete gamma function $\Gamma(\kappa, x)$ is defined by [39]

$$\Gamma(\kappa, x) := \int_x^\infty \mathrm{d}t\, \mathrm{e}^{-t} t^{\kappa-1} \tag{25}$$

and has the following asymptotic behaviour for large or small arguments

$$\Gamma(\kappa, x) \stackrel{|x|\gg 1}{\approx} x^{\kappa-1}\,\mathrm{e}^{-x}, \qquad \Gamma(\kappa, x) \stackrel{0<x\ll 1}{\approx} \Gamma(\kappa) - \frac{x^\kappa}{\kappa}. \tag{26}$$

In the limit where both s and $t - s$ become large, we recover equation (19) as it should be. Furthermore, we explicitly see that the dynamical exponent $z = 2$.

For illustration, we have also evaluated the integral (17) numerically (with $\alpha\rho_0 = 1$). In figure 2(*a*) we compare the numerical results, for several values of s in three dimensions, with the analytical result (19). We see that already for quite small values of s one has a nice data collapse which confirms the expected scaling behaviour. Furthermore, the agreement with the analytically calculated scaling function is perfect. In figure 2(*b*) we display the dependence on $\boldsymbol{r}$, evaluated along the line $\boldsymbol{r} = (r, 0, \ldots, 0)$. Again, the expected scaling behaviour is also confirmed and the curves agree with the analytical expression (24).

In the non-critical case, we have for the density, after rescaling $t \to t/(2D)$

$$\frac{\partial}{\partial t}\rho(\boldsymbol{x}, t) = \frac{1}{2}\Delta_{\boldsymbol{x}}\rho(\boldsymbol{x}, t) + \frac{1}{2}\eta\rho(\boldsymbol{x}, t) \qquad \text{with} \quad \eta := \frac{\mu k - \lambda\ell}{D}. \tag{27}$$

This is easily solved and yields

$$\rho(\boldsymbol{x}, t) = \rho_0\, \mathrm{e}^{\frac{1}{2}\eta t} \tag{28}$$

if we choose again a homogenous initial distribution with mean density ρ_0. Depending on whether particle creation or annihilation dominates, the density increases or decreases exponentially. Next, for the single-time correlator $F(\boldsymbol{r}, t)$, we use [26, equations (7) and (8)] which can be written after rescaling as (recall $\ell = 1$)

$$\frac{\partial}{\partial t}F(\boldsymbol{r}, t) = \Delta F(\boldsymbol{r}, t) + \eta F(\boldsymbol{r}, t) + \alpha\delta_{\boldsymbol{r},0}\rho(t) \tag{29}$$

which is easily solved by introducing the particle density (28) and performing a Fourier transform. This in turn allows us to solve the equation of motion (14) for the two-time correlator and we find

$$F(\boldsymbol{r};t,s) = \rho_0^2\,\mathrm{e}^{\frac{1}{2}\eta(t+s)} + \alpha\rho_0\,\mathrm{e}^{\frac{1}{2}\eta(t+s)}\int_0^s \mathrm{d}\tau\,\mathrm{e}^{-\frac{1}{2}\eta\tau}\,b\left(\boldsymbol{r};\frac{1}{2}(t+s)-\tau\right). \quad (30)$$

We consider the case $\boldsymbol{r}=0$ and t and s in the ageing regime. As before, we use the asymptotic expression for $b\big(\mathbf{0},\frac{1}{2}(t+s)-\tau\big)$ and find for the connected autocorrelator

$$\begin{aligned} G(\mathbf{0};t,s) &:= F(\mathbf{0};t,s) - \rho_0^2\,\mathrm{e}^{\frac{1}{2}\eta(t+s)} \\ &= \frac{\alpha\rho_0\,\mathrm{e}^{\frac{1}{4}\eta(t+s)}}{(4\pi)^{d/2}}\left[\Gamma\left(-\frac{d}{2}+1,-\frac{\eta}{4}(t+s)\right)-\Gamma\left(-\frac{d}{2}+1,-\frac{\eta}{4}(t-s)\right)\right]. \end{aligned} \quad (31)$$

Using the asymptotic behaviour (26) for the gamma function for large arguments we obtain

$$G(\mathbf{0};t,s) = -\frac{2\alpha\rho_0}{(2\pi)^{d/2}\eta}\left[(t-s)^{-d/2}\exp\left(\frac{\eta}{2}t\right)-(t+s)^{-d/2}\exp\left(\frac{\eta}{2}(t+s)\right)\right]. \quad (32)$$

If η is positive, then particle creation outweighs particle annihilation. The second term dominates and leads to an exponential divergence. On the other hand, if η is negative, the first term involving $\mathrm{e}^{\eta t/2}$ is the dominant one. At first sight, these results appear curious since the leading exponential behaviour merely depends on $t+s$ and t, respectively, and not on $t-s$, as might have been anticipated.

A similar result had already been found in the inactive phase of the ordinary contact process [23, 24] and we can understand the present result along similar lines. Consider the limits where $|\eta|\to\infty$, such that diffusion plays virtually no role in comparison with the creation or annihilation processes. Then merely the creation and annihilation processes on a single site need to be considered. Correlators are given in terms of conditional probabilities and we now consider the two cases $\eta>0$ and $\eta<0$. First, for $\eta<0$, annihilation dominates and at late times there are only a few particles left in the system. This is the same situation as in the inactive phase of the ordinary contact process. Then $G(\mathbf{0};t,s)$ can only be non-vanishing if at time s a particle was present and it should only depend on t. On the other hand, for $\eta>0$, the particle density diverges exponentially and the number of possible reactions is conditioned by the density at time s, proportional to $\mathrm{e}^{\eta s/2}$, hence the dependence on $t+s$. Finally, the power-law prefactors relate to the diffusion between different sites.

4. The bosonic critical pair-contact process

For the bosonic pair-contact process, we have $p=m=2$. System (14) of differential equations closes only for the critical case, i.e. for $\lambda\ell=\mu k$, and we shall restrict to this situation throughout. At criticality, the values of ℓ and k do not influence the scaling behaviour. It was shown in [26] that in dimensions $d>2$ there is a phase transition along the critical line and we must therefore distinguish three cases, according to whether the reduced control parameter

$$\alpha' := \frac{\alpha-\alpha_C}{\alpha_C} \quad (33)$$

is negative, zero or positive and where α was defined in (12) and α_C in (13). For $d\leqslant 2$ one is always in the situation $\alpha'>0$. We recall the known results for the single-time autocorrelator $F(\mathbf{0},t)$ which for large times behaves as [26]

- $\alpha < \alpha_C$.

$$F(\mathbf{0}, t) \stackrel{t\to\infty}{\approx} -\frac{\rho_0^2}{\alpha'}. \tag{34}$$

- $\alpha = \alpha_C$.

$$F(\mathbf{0}, t) \stackrel{t\to\infty}{\approx} \begin{cases} \dfrac{(4\pi)^{\frac{d}{2}}\rho_0^2}{|\Gamma(1-d/2)|\alpha_C} t^{\frac{d}{2}-1} & \text{for} \quad 2 < d < 4 \\ \dfrac{\rho_0^2}{4A_2\alpha_C} t & \text{for} \quad d > 4 \end{cases} \tag{35}$$

where A_2 is a known constant which is defined in [26].

- $\alpha > \alpha_C$ or $d < 2$.

$$F(\mathbf{0}, t) \stackrel{t\to\infty}{\approx} A\rho_0^2 \exp(t/\tau_{\mathrm{ts}}). \tag{36}$$

The known prefactor A and the time scale τ_{ts} are dimension dependent and positive. The exact expressions for them are not essential for our considerations and can be found in [26].

The solution of the equations of motion is quite analogous to the one of the bosonic contact process and the results from section 3 can be largely taken over. We find, again for initially uncorrelated particles of mean density ρ_0,

$$F(\boldsymbol{r}; t, s) = \rho_0^2 + \alpha \int_0^s \mathrm{d}\tau\, F(\mathbf{0}, \tau) b\left(\boldsymbol{r}, \frac{1}{2}(t+s) - \tau\right). \tag{37}$$

For $t = s$ this formula agrees with [26, equation (21)] as it should. We are interested in the behaviour of the connected correlation function, see (17), in the ageing regime. The analysis of equation (37) is greatly simplified by recognizing that, quite in analogy with ageing in simple ferromagnets, there is some intermediate time scale t_p such that for times $\tau \lesssim t_p$, one is still in some quasi-stationary regime while for $\tau \gtrsim t_p$ one goes over into the ageing regime and that furthermore, the cross-over between these regimes occurs very rapidly [43]. We denote by $F_{\mathrm{age}}(\mathbf{0}, \tau)$ the asymptotic ageing form of $F(\mathbf{0}, \tau)$ and write

$$\begin{aligned} \int_0^s \mathrm{d}\tau\, F(\mathbf{0}, \tau) b\left(\mathbf{0}, \frac{1}{2}(t+s) - \tau\right) &= \int_0^{t_p} \mathrm{d}\tau\, F(\mathbf{0}, \tau) b\left(\mathbf{0}, \frac{1}{2}(t+s) - \tau\right) \\ &\quad + s\int_{t_p/s}^1 \mathrm{d}v\, F_{\mathrm{age}}(\mathbf{0}, sv) b\left(\mathbf{0}, \frac{1}{2}(t+s) - \tau v\right). \end{aligned} \tag{38}$$

We denote the first term of the last line by $C_1(t, s, t_p)$. Since we expect that $t_p \sim s^\zeta$ with $0 < \zeta < 1$ [43], we can replace the lower integration limit by 0 in the second integral. This leaves us with the result

$$G(\mathbf{0}; t, s) = C_1(t, s, t_p) + \int_0^s \mathrm{d}\tau\, F_{\mathrm{age}}(\mathbf{0}, \tau) b\left(\mathbf{0}, \frac{1}{2}(t+s) - \tau\right). \tag{39}$$

On the other hand, we have the following rough estimate

$$\begin{aligned} |C_1(t, s, t_p)| &\leqslant t_p \max_{\tau\in[0,t_p]} \left| F(\mathbf{0}, \tau) b\left(\mathbf{0}, \frac{1}{2}(t+s) - \tau\right)\right| \\ &\stackrel{s\gg 1}{\approx} t_p \max_{\tau\in[0,t_p]} |F(\mathbf{0}, \tau)| s^{-\frac{d}{2}} \left(4\pi\left(\frac{1}{2}(t/s+1) - \frac{t_p}{s}\right)\right)^{-\frac{d}{2}}. \end{aligned} \tag{40}$$

In the three cases (i) $\alpha < \alpha_C$ and $d > 2$, (ii) $\alpha = \alpha_C$ and $2 < d < 4$ and (iii) $\alpha = \alpha_C$ and $d > 4$ this leads by equations (34) and (35), respectively, to the upper bounds $|C_1| \lesssim s^{\zeta - d/2}$, $s^{(\zeta-1)d/2}$ and $s^{2\zeta - d/2}$ which vanish for s large more rapidly than $G(\mathbf{0}; t, s) \sim s^{1-d/2}$, s^0 and $s^{2-d/2}$, respectively and which are derived below. Hence $C_1(t, s, t_p)$ is irrelevant for the determination of b and the scaling functions and will be dropped in what follows. Similarly, because of (36), $C_1(t, s, t_p)$ is non-leading if $\alpha > \alpha_C$.

We have also checked that for $d > 4$ this same result can be derived more explicitly using a Laplace transformation, along the lines of [26]. For the sake of brevity, these relatively straightforward calculations will not be reproduced here [41].

4.1. Ageing regime: $r = 0$ *and* $s, t - s \gg 1$

The most interesting cases are $d > 2$ and $\alpha \leqslant \alpha_C$, which we will treat first. The asymptotic expression for $F(\mathbf{0}, t)$ is of the form $F_{\rm age}(\mathbf{0}, t) = \mathcal{A}\rho_0^2 t^{\xi}$, where ξ and the prefactor $\mathcal{A}$ can be read from equations (34) and (35).

We therefore get for the connected autocorrelator

$$
\begin{aligned}
G(\mathbf{0}; t, s) &= \frac{\alpha\rho_0^2\mathcal{A}}{(4\pi)^{\frac{d}{2}}} \int_0^s \mathrm{d}\tau\, \tau^{\xi} \left(\frac{1}{2}(t+s) - \tau\right)^{-\frac{d}{2}} \\
&= \frac{\alpha\rho_0^2\mathcal{A}}{(\xi+1)(4\pi)^{\frac{d}{2}}} s^{\xi+1-\frac{d}{2}} \left(\frac{1}{2}(y+1)\right)^{-\frac{d}{2}} {}_2F_1\left(\frac{d}{2}, \xi+1; \xi+2; \frac{2}{y+1}\right)
\end{aligned} \tag{41}
$$

where $y = t/s$ and ${}_2F_1$ is a hypergeometric function. We deduce the general form of the scaling function

$$
f_G(y) = \frac{\alpha\rho_0^2\mathcal{A}}{(\xi+1)(4\pi)^{\frac{d}{2}}} \left(\frac{1}{2}(y+1)\right)^{-\frac{d}{2}} {}_2F_1\left(\frac{d}{2}, \xi+1; \xi+2; \frac{2}{y+1}\right) \tag{42}
$$

and the exponents

$$
b = -\xi - 1 + \frac{d}{2} \qquad \text{and} \qquad \lambda_G = d \tag{43}
$$

and furthermore $z = 2$, see [26] and below. For the different cases we obtain the following explicit expressions.

- *$\alpha < \alpha_C$ and $d > 2$.* Here we have $\xi = 0$ and the prefactor is $\mathcal{A} = -\frac{1}{\alpha'}$. Therefore, we have a value of

$$
b = \frac{d}{2} - 1. \tag{44}
$$

The ${}_2F_1$ function can be rewritten with the help of the relation (9.121,5) from [42] so that we obtain an elementary expression for the scaling function

$$
f_G(y) = \frac{\rho_0^2\alpha}{\alpha'(2\pi)^{\frac{d}{2}}(d-2)} \left((y+1)^{-\frac{d}{2}+1} - (y-1)^{-\frac{d}{2}+1}\right). \tag{45}
$$

- *$\alpha = \alpha_C$ and $d > 2$.* Here we have $\xi = \frac{d}{2} - 1$ and $\xi = 1$ for $2 < d < 4$ and $d > 4$ respectively. This implies for b

$$
b = \begin{cases} 0 & \text{for} \quad 2 < d < 4 \\ \dfrac{d}{2} - 2 & \text{for} \quad d > 4. \end{cases} \tag{46}
$$

For $2 < d < 4$ the scaling function is (with the prefactor $\mathcal{A} = \frac{(4\pi)^{\frac{d}{2}}}{\left|\Gamma\left(1-\frac{d}{2}\right)\right|\alpha_C}$ [26])

$$f_G(y) = \frac{2^{\frac{d}{2}+1}\rho_0^2}{d\left|\Gamma\left(1-\frac{d}{2}\right)\right|}(y+1)^{-\frac{d}{2}}{}_2F_1\left(\frac{d}{2}, \frac{d}{2}; \frac{d}{2}+1; \frac{2}{y+1}\right). \tag{47}$$

For $d > 4$, the scaling function (42) can again be written as an elementary function with the help of a Gauß recursion relation (equation (9.137,4) from [42])

$$f_G(y) = \frac{\rho_0^2}{4A_2(2\pi)^{\frac{d}{2}}(d-2)(d-4)} \times \left((y+1)^{-\frac{d}{2}+2} - (y-1)^{-\frac{d}{2}+2} + (d-4)(y-1)^{-\frac{d}{2}+1}\right). \tag{48}$$

- *$\alpha > \alpha_C$ or $d < 2$.* Due to the exponential behaviour of $F(\mathbf{0}, \tau)$ we do not have scaling behaviour in these cases. The integrals which enter the calculation are similar to those encountered in (31) and where the time scale τ_{ts} and the factor A are defined in equation (36):

$$G(\mathbf{0}; t, s) = \frac{\alpha\rho_0^2 A\, \mathrm{e}^{t+s/(2\tau)}}{(4\pi)^{d/2}}\tau_{\text{ts}}^{\frac{d}{2}}\left[\Gamma\left(-\frac{d}{2}+1, \frac{t+s}{2\tau_{\text{ts}}}\right) - \Gamma\left(-\frac{d}{2}+1, \frac{t-s}{2\tau_{\text{ts}}}\right)\right]. \tag{49}$$

Using the asymptotic behaviour of the gamma function (26), we see that the leading term in the scaling limit is

$$G(\mathbf{0}; t, s) \simeq \frac{\alpha\rho_0^2 A}{(2\pi)^{\frac{d}{2}}}(t-s)^{-d/2}\exp\frac{s}{\tau_{\text{ts}}}. \tag{50}$$

In contrast with the other cases treated before, the connected autocorrelator *increases* exponentially with the waiting time s.

4.2. $\boldsymbol{r}$ dependence for $s, t \gg 1$

In order to compute the $\boldsymbol{r}$ dependence of the correlator, we follow the same strategy as in the last section. We use approximation (23) which can be justified by an argument relying on an inequality similar to (40). We obtain the following results.

- *$\alpha < \alpha_C$ and $d > 2$.* As we have $F(\mathbf{0}, \tau) \approx -\rho_0^2/\alpha'$ the computation is the same as for the contact process, compare equation (24). The result is

$$G(\boldsymbol{r}; t, s) = \frac{-\alpha\rho_0^2}{(4\pi)^{\frac{d}{2}}\alpha'}\left(\frac{\boldsymbol{r}^2}{4}\right)^{-(\frac{d}{2}-1)}\left[\Gamma\left(\frac{d}{2}-1, \frac{\boldsymbol{r}^2}{2(t+s)}\right) - \Gamma\left(\frac{d}{2}-1, \frac{\boldsymbol{r}^2}{2(t-s)}\right)\right]. \tag{51}$$

- *$\alpha = \alpha_C$ and $d > 4$.* We find the following result

$$G(\boldsymbol{r}; t, s) = \frac{\rho_0^2}{4A_2(4\pi)^{\frac{d}{2}}}\left(\frac{\boldsymbol{r}^2}{4}\right)^{-(\frac{d}{2}-1)} \times \left[\frac{t+s}{2}\left(\Gamma\left(\frac{d}{2}-1, \frac{\boldsymbol{r}^2}{2(t+s)}\right) - \Gamma\left(\frac{d}{2}-1, \frac{\boldsymbol{r}^2}{2(t-s)}\right)\right) - \left(\frac{\boldsymbol{r}^2}{4}\right)\left(\Gamma\left(\frac{d}{2}-2, \frac{\boldsymbol{r}^2}{2(t+s)}\right) - \Gamma\left(\frac{d}{2}-2, \frac{\boldsymbol{r}^2}{2(t-s)}\right)\right)\right]. \tag{52}$$

It is straightforward to check consistency with (48) for the case s and $t-s$ much larger than $\boldsymbol{r}^2$ by using the asymptotic form (26) of the gamma function.

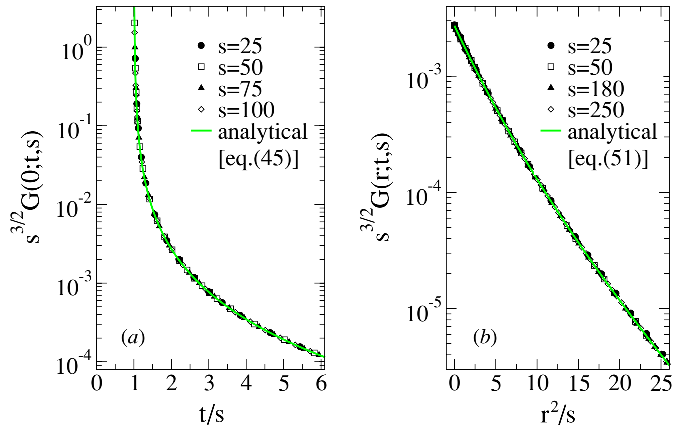

Figure 3. Scaling plots for the case $\alpha' < 0$ of (*a*) the autocorrelation function $G(0; t, s)$ and (*b*) the space-dependent correlation function $G(\boldsymbol{r}; t, s)$ for the bosonic pair-contact process in five dimensions and with $\alpha\rho_0 = 1$. In (*b*), the value of $y = t/s = 2$ was used.

- $\alpha = \alpha_C$ *and* $2 < d < 4$.

$$G(\boldsymbol{r}; t, s) = \frac{\rho_0^2}{(d/2)\left|\Gamma\left(\frac{d}{2}-1\right)\right|} \int_0^s \mathrm{d}\tau\, \tau^{\frac{d}{2}-1} \left(\frac{1}{2}(t+s) - \tau\right)^{-\frac{d}{2}} \exp\left(-\frac{\boldsymbol{r}^2}{2(t+s-2\tau)}\right).$$

We now develop the exponential function. The integrals are similar to those already seen so that we merely state the result

$$G(\boldsymbol{r}; t, s) = \frac{2\rho_0^2}{d\left|\Gamma\left(\frac{d}{2}-1\right)\right|} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\boldsymbol{r}^2}{4s}\right)^n \left(\frac{t/s+1}{2}\right)^{-\frac{d}{2}-n} \times {}_2F_1\left(\frac{d}{2}+n, \frac{d}{2}; \frac{d}{2}+1; \frac{2}{t/s+1}\right). \tag{53}$$

- $\alpha > \alpha_C$ *or* $d < 2$. Here again we develop the exponential function and obtain as the final result

$$G(\boldsymbol{r}; t, s) = \frac{\alpha\rho_0^2 A}{(4\pi)^{\frac{d}{2}}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\boldsymbol{r}^2}{4}\right)^n \exp\left(\frac{t+s}{2\tau_{\mathrm{ts}}}\right) \tau_{\mathrm{ts}}^{\frac{d}{2}+n} \times \left(\Gamma\left(-\frac{d}{2}-n+1, \frac{t+s}{2\tau_{\mathrm{ts}}}\right) - \Gamma\left(-\frac{d}{2}-n+1, \frac{t-s}{2\tau_{\mathrm{ts}}}\right)\right). \tag{54}$$

In view of the numerous approximations needed to derive these results, it is of interest to check them numerically. In figure 3, we compare the results of the numerical integration of (37) with the analytical predictions (44), (45) and (51) which apply for $\alpha < \alpha_C$ and $d = 5$. The nice collapse of the data shows that the scaling regime is already reached for the relatively small values of s used. The perfect agreement of the data with the analytical results confirms that dropping the term C_1 in (39) is justified (and suggests that C_1 should be considerably smaller than the rough estimate (40)). Similarly, we compare data for $\alpha = \alpha_C$ in 5D with the predictions (48) and (52) in figure 4 and similarly in 3D in figure 5. Again the agreement is perfect.

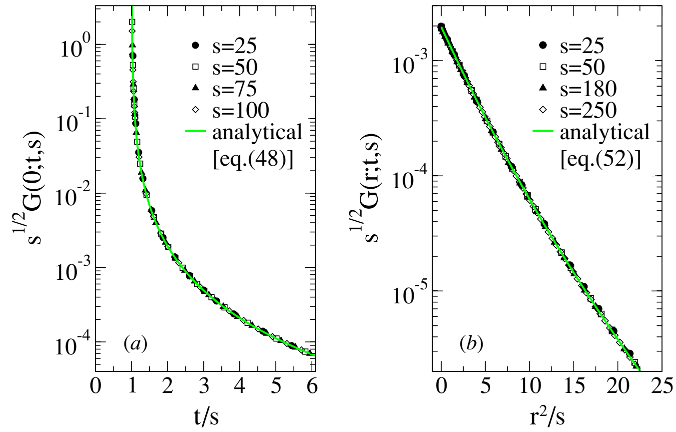

Figure 4. Scaling plots for the case $\alpha' = 0$ of (a) the autocorrelation function $G(0; t, s)$ and (b) the space-dependent correlation function $G(\boldsymbol{r}; t, s)$ for the bosonic pair-contact process in five dimensions and with $\alpha\rho_0 = 1$. In (b) the value of $y = t/s = 2$ was used.

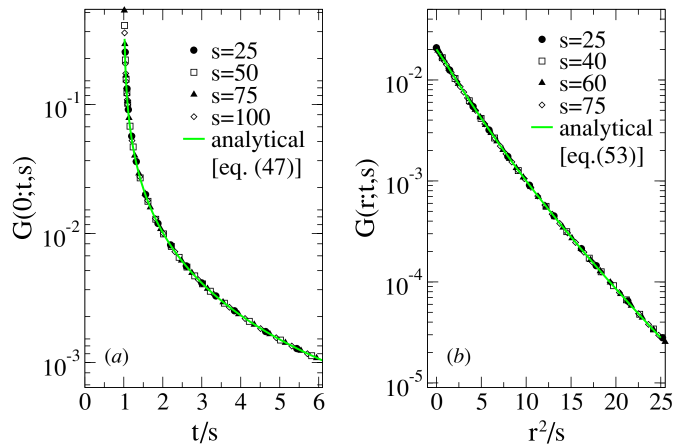

Figure 5. Scaling behaviour of the two-time correlator G for the case $\alpha' = 0$ and three dimensions. The value $\alpha\rho_0$ was set to unity. In (b) the value of $y = t/s = 2$ was used. The data collapse occurs for $b = 0$.

5. Response functions

The response function of the first moment to an external field $h(\boldsymbol{y}, s)$ is given by

$$R(\boldsymbol{x}, \boldsymbol{y}; t, s) := \left. \frac{\delta \langle a(\boldsymbol{x}, t) \rangle}{\delta h(\boldsymbol{y}, s)} \right|_{h=0}. \tag{55}$$

5.1. The contact process

We apply definition (55) on both sides of the equation of motion (27) and find, exploiting spatial translation invariance, with $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$

$$\frac{\partial}{\partial t} R(\boldsymbol{r}; t, s) = \frac{1}{2} \Delta R(\boldsymbol{r}; t, s) + \frac{1}{2} \eta R(\boldsymbol{r}; t, s) + \delta(t - s). \tag{56}$$

Table 1. Ageing exponents of the critical bosonic contact and pair-contact processes in the different regimes. The results for the bosonic contact process hold for an arbitrary dimension d, but for the bosonic pair-contact process they only apply if $d > 2$ since $\alpha_C = 0$ for $d \leqslant 2$.

	Bosonic contact process	Bosonic pair-contact process		
		$\alpha < \alpha_C$	$\alpha = \alpha_C$	
a	$\frac{d}{2}-1$	$\frac{d}{2}-1$	$\frac{d}{2}-1$	
b	$\frac{d}{2}-1$	$\frac{d}{2}-1$	0	if $2 < d < 4$
			$\frac{d}{2}-2$	if $d > 4$
λ_R	d	d	d	
λ_G	d	d	d	
z	2	2	2	

This is the defining equation of a diffusion-type Green function with the solution

$$R(\boldsymbol{r}; t, s) = r_0\, \mathrm{e}^{\frac{1}{2}\eta(t-s)} b\left(\boldsymbol{r}, \tfrac{1}{2}(t-s)\right) \Theta(t-s) \tag{57}$$

where $b(\boldsymbol{r}, t)$ was given in equation (18) and r_0 is a normalization constant. This expression is invariant under time translations and remains so even at criticality[5].

5.2. *The critical pair-contact process*

The equation of motion for the particle density on the critical line does not change in comparison with the contact process, so that we can take over result (57) with η set to zero and have

$$R(\boldsymbol{r}; t, s) = r_0 b\left(\boldsymbol{r}, \frac{1}{2}(t-s)\right) \Theta(t-s). \tag{58}$$

The autoresponse function in the scaling regime is obtained by setting $\boldsymbol{r} = 0$ and using the known asymptotic behaviour of the Bessel function, with the result $(t > s)$

$$R(t, s) := R(\boldsymbol{0}; t, s) \simeq r_0(2\pi(t-s))^{-d/2} \tag{59}$$

from which we can read off the scaling function and the exponents a and λ_R

$$a = \frac{d}{2} - 1, \qquad f_R(y) = \frac{r_0}{(2\pi)^{d/2}}(y-1)^{-d/2}, \qquad \lambda_R = d. \tag{60}$$

We collect our results for the ageing exponents $a, b, \lambda_G, \lambda_R, z$ in table 1. A few comments are now in order. First, for both the critical bosonic contact process and the critical bosonic pair-contact process with $\alpha < \alpha_C$, we see by comparing the result for a with the corresponding ones for b, see table 1, that $a = b$. Together with the identity $\lambda_G = \lambda_R$, the critical ageing behaviour of these systems is quite analogous to that of simple, reversible ferromagnets quenched to their critical temperature. Second, the critical bosonic pair-contact process with $\alpha = \alpha_C$ furnishes an analytically solved example where a and b are different. This is analogous to the result found for the 1D and 2D critical ordinary contact processes, where $a = b - 1$ was observed [23, 24] and where the relation $\lambda_G = \lambda_R$ holds as well. However, there is no apparent simple and

[5] Ageing is characterized by the existence of several competing stable stationary states (or a critical point) and time-translation invariance (TTI) can no longer be *requested*. However, that does not mean that TTI were always impossible and indeed TTI can be recovered as a limit case, for certain specific values of the ageing exponents. A well-known example is the the response function of the spherical model quenched onto criticality ($T = T_c$) in $d > 4$ space dimensions [5].

general relation between the exponents a and b for ageing systems without detailed balance. Third, our results for the critical bosonic pair-contact process provide further evidence against the generality of a recent proposal by Sastre *et al* [44] to define a non-equilibrium temperature which was based on the implicit assumption that $a = b$ would remain true even in the absence of detailed balance. Fourth, we can compare the form of the scaling function $f_R(y)$ of the autoresponse with the prediction of local scale invariance quoted in equation (5). We find perfect agreement and identify $a = a'$. Fifth, we recall that for $z = 2$ there is a variant of local scale invariance which takes the presence of a discrete lattice into account. It is possible to construct the corresponding representation of the Schrödinger Lie-algebra and then a response function transforming covariantly under it should read for $t > s$ in d spatial dimensions [45]

$$R(\boldsymbol{r}; t, s) = r_0(t-s)^{(d-2x)/2} \exp\left(\frac{d(t-s)}{\mathcal{M}}\right) \mathcal{I}_{\boldsymbol{r}}\left(\frac{t-s}{\mathcal{M}}\right), \qquad \mathcal{I}_{\boldsymbol{r}}(u) := \prod_{j=1}^{d} I_{r_j}(u) \tag{61}$$

where x is a scaling dimension and r_0, $\mathcal{M}$ are constants. Here the spatial distance $\boldsymbol{r}$ is an integer multiple of the lattice constant. Comparison with equations (58) and (18) shows complete agreement if we identify $x = d/2$ and $\mathcal{M} = 1/2$.

6. Conclusions

We have studied the ageing behaviour of the exactly solvable bosonic contact process and of the bosonic critical pair-contact process in order to get a better understanding of how the present scaling description of ageing, which is derived from the study of reversible systems with detailed balance, should be generalized for truly irreversible systems without detailed balance. This more general situation might be closer to what is going on in chemical or biological ageing than the reversible systems undergoing physical ageing, e.g. after a temperature quench. In comparison with the ordinary contact and pair-contact processes, these bosonic models permit an accumulation of many particles on a single site and this possibility does indeed affect the long-time behaviour of these models. Trivially, if either particle production or annihilation dominates, the mean occupation number will either diverge for large times or the population will die out, but if these rates are balanced there is a critical line where the mean particle density is constant in time and the system's behaviour is more subtle. Indeed, on the critical line the long-time behaviour depends on how effectively single-particle diffusion is capable of homogenizing the system, see figure 1. For dimensions $d \leqslant 2$, there is always *clustering* at criticality, that is a few sites are highly populated and the others are empty. On the other hand, for $d > 2$ there is no clustering in the bosonic contact process, but in the bosonic pair-contact process there is a *clustering transition* at some $\alpha = \alpha_C$ such that clustering occurs for $\alpha > \alpha_C$ (where the diffusion is relatively weak) and there is a more or less homogeneous state for $\alpha \leqslant \alpha_C$.

This behaviour of the models is also reflected in their ageing behaviour which we studied here. We anticipated in the ageing regime $t, s \gg 1$ and $t - s \gg 1$ the scaling forms for the connected autocorrelator and autoresponse

$$G(t, s) := G(\mathbf{0}; t, s) = s^{-b} f_G(t/s), \qquad R(t, s) := R(\mathbf{0}; t, s) = s^{-1-a} f_R(t/s) \tag{62}$$

together with the asymptotics $f_{G,R}(y) \sim y^{-\lambda_{G,R}/z}$ as $y \gg 1$ and our results for the exponents and the scaling functions are listed in tables 1 and 2. Specifically,

(i) For $d > 2$, the ageing of the bosonic pair-contact process for $\alpha < \alpha_C$ lies in the same universality class as the bosonic contact process since all critical exponents and the scaling functions coïncide. Furthermore, the ageing behaviour in the bosonic contact and pair-contact processes does not depend on whether the parity of the total number of particles

Table 2. Scaling functions of the autoresponse and autocorrelation of the critical bosonic contact and bosonic pair-contact processes. They are only given up to a multiplicative factor, which may depend on the dimension. The logarithmic form (22) of $f_G(y)$ for the 2D bosonic contact process may be obtained from a $d \to 2$ limit.

			$f_R(y)$	$f_G(y)$
Contact process			$(y-1)^{-\frac{d}{2}}$	$(y-1)^{-\frac{d}{2}+1} - (y+1)^{-\frac{d}{2}+1}$
Pair-contact process	$\alpha < \alpha_C$	$d > 2$	$(y-1)^{-\frac{d}{2}}$	$(y-1)^{-\frac{d}{2}+1} - (y+1)^{-\frac{d}{2}+1}$
		$2 < d < 4$	$(y-1)^{-\frac{d}{2}}$	$(y+1)^{-\frac{d}{2}}{}_2F_1\left(\frac{d}{2}, \frac{d}{2}; \frac{d}{2}+1; \frac{2}{y+1}\right)$
	$\alpha = \alpha_C$	$d > 4$	$(y-1)^{-\frac{d}{2}}$	$(y+1)^{-\frac{d}{2}+2} - (y-1)^{-\frac{d}{2}+2} + (d-4)(y-1)^{-\frac{d}{2}+1}$

is conserved or not. All these systems have in common that their behaviour is strongly influenced by single-particle diffusion. One might wonder whether an analogy to the Janssen–Grassberger conjecture [46, 47] could be formulated[6].

(ii) While for $d < 2$, we still find dynamical scaling behaviour in the critical bosonic contact process, there is no such scaling for the bosonic pair-contact process if $\alpha > \alpha_C$, hence in particular for $d \leqslant 2$. Therefore, although both models have the same topology of their phase diagrams for $d < 2$, see figure 1(*a*), their ageing behaviour is different.

(iii) At the clustering transition $\alpha = \alpha_C$ in the critical bosonic pair-contact process, dynamical scaling occurs, but the ageing exponents a and b are different. Here the absence of detailed balance leads to a substantial modification of the scaling description with respect to what happens in critical ferromagnets. In particular, there is no non-trivial analogue of the limit fluctuation–dissipation ratio of critical ageing ferromagnets. A relation $a \neq b$, see (6), has also been observed in the ordinary critical contact process which also shares the property that $\lambda_G = \lambda_R$ still holds [23, 24]. However, according to the known examples, a simple and general relation between a and b does not seem to exist for systems without detailed balance. Further evidence from other non-equilibrium models would be welcome.

(iv) On the other hand, the equality $\lambda_G = \lambda_R$ between the autocorrelation and autoresponse exponents, at the critical point of the steady state and for uncorrelated initial states, seems to be a generic feature even for systems without detailed balance.

(v) The form of the response function is in full agreement with local scale invariance which confirms that the annihilation operator $a(\boldsymbol{x})$ is a suitable candidate for a quasi-primary field[7] of local scale invariance. We shall come back to a detailed analysis of the correlators from the point of view of local scale invariance in a following paper.

Explicit results were also derived for the space-dependent scaling functions of spacetime correlator and responses. For the contact process, the space-dependent response function is given by equation (57) and the space-dependent correlation function by equation (24). For the critical pair-contact process, the space-dependent response function is given by equation (58). The space-dependent correlation function can be found in (i) equation (51) for the case $\alpha < \alpha_C$ and $d > 2$, in (ii) equation (52) for the case $\alpha = \alpha_C$ and $d > 4$,

[6] An important ingredient of the models studied here seems to be that at criticality the mean particle density stays constant. On the other hand, even if a 'soft' limit on the particle number per site is introduced, e.g. by a further reaction $3A \to 2A$, the long-time behaviour is likely to be that of the PCPD, as checked for the particle density in [48].

[7] In conformal field-theory, a quasi-primary field transforms covariantly under the action of the conformal group [49]. This concept can be generalized to fields transforming covariantly under the action of a group of local scale transformations, see [15] and references therein for details.

in (iii) equation (53) for the case $\alpha = \alpha_C$ and $2 < d < 4$ and in (iv) equation (54) for the case $\alpha > \alpha_C$ or $d > 2$.

Finally, we comment on a suggested relationship between the bosonic pair-contact process and the spherical model [26]. In the spherical model, a classical result by Berlin and Kac [50] states that the magnetization is spatially uniform; in particular, the possibility that almost the entire macroscopic magnetization was carried by a single spin can be excluded. This is in remarkable contrast to the clustering transition which occurs in the bosonic pair-contact process. More formally, a closer inspection shows notable differences between the spherical constraint and the analogous equation used to derive the correlator $F(\mathbf{0}, t)$. This suggests that the analogies between the two models do not seem to have a deeper physical basis.

Acknowledgments

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